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COMBINED DRY FRICTION MODELS IN THE CASE OF RANDOM DISTRIBUTION OF THE NORMAL CONTACT STRESSES INSIDE CONTACT PATCHES

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Abstract. We propose a further development of the theory of multi-component dry friction [5-10] by offering a dry friction model with random distribution of contact stresses across the contact spot. Special attention is devoted to investigation of the dry friction effects that arise in the traditional systems that consist of solid disks and spheres sliding or rolling on a horizontal plane. Applications of the proposed model for description of the interaction between a pneumatic aviation tire and the surface of a landing strip are outlined.

1 INTRODUCTION

The theory of poly-component dry friction has proved its efficiency in the investigation of the dynamics of solids for a variety of types of kinematic interaction between them. The theory allowed a relevant description of the overall effects that the poly-component friction can produce and also helped create physically consistent phenomenological models of friction.

Tackling real engineer problems within the framework of the theory (such as rolling of aviation pneumatic tires) revealed that the model must be further improved by taking into account the environmental and physical realities which in its turn seriously affect the value of the coefficients involved. In particular, the anisotropy of the dry friction coefficient should be taken into consideration [9, 18].

Another factor that influences the coefficients of the model is the non-homogeneity of the spot across which the bodies interact which leads to a certain randomness in the distribution of the contact stresses within the spot. This phenomenon of “randomness” shows itself not only in real-life problems [2-4, 16,17] but also in model experiments [11] aimed at verification of the theory we offer. First attempts to account for this “randomness” are due to

Burlakov and Treschev [1] where they suggested a simplified contact stress model (e.g. a rigid circle is assumed to touch the plane in a three randomly emerging supporting points). Below we present a model in which the normal contact stresses contain some randomly distributed components. Such an approach seems to be relevant for the study of an amount of classical problems such as sliding and spinning of a massive disk (cylinder) and rolling accompanied with slipping and spinning of a heavy sphere over a rough surface. In spite of its seeming simplicity this approach serves a good and convenient basis for further more sophisticated analysis.

2 PROBABILISTIC MODEL OF SLIDING FRICTION

2.1 Basic assumptions

Consider a system that consists of two rigid bodies that interact frictionally. A complicated (combined) kinematics within the contact area is assumed meaning that any combination of sliding, spinning and rolling may occur (Fig. 1). The contact spot is assumed to be of circular (or nearly circular) shape. It was shown in the previous studies [5-14] that the spot's shape seriously influences the net force that the bodies experience and especially the occurrence of the essentially non-zero dry friction component $F_{\perp}$, which is orthogonal to the velocity of the relative sliding. Nevertheless, if the spot is almost a circle this component exerts but a negligible effect on the $F_{\|}$ component, which opposes the relative sliding velocity.

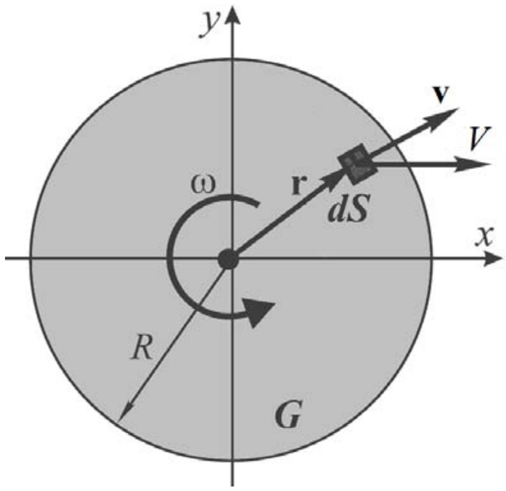

Figure 1: Kinematics inside the contact spot

The dry friction models describing the interaction within a contact spot are constructed under the assumption that the Coulomb law in generalized differential form holds for a small surface element dS in the interior of the contact spot. According to the law the differentials of the resultant vector $d\mathbf{F}$ and the moment of friction dM_C with respect to the contact spot center are determined by the formulae:

$$d\mathbf{F} = -f\sigma\frac{\mathbf{V}}{|\mathbf{V}|}dS,\, dM_C = -f\sigma\frac{\mathbf{r}\times\mathbf{V}}{|\mathbf{V}|}dS, \qquad (1)$$
$$\mathbf{V} = (v - \omega y, \omega x),\, \mathbf{r} = (x, y),\, \sigma = \sigma(x, y)$$

Here f is the coefficient of friction, $\mathbf{r} = (x, y)$ is the position vector of an elemental area in the interior of the contact spot with respect to its center Fig. 1, ω is the angular velocity of rotation about the center and $\sigma(x, y)$ is the distribution of the normal contact stresses.

Formulas (1) clearly manifest the following important feature: the distribution of velocities and normal contact stresses within the spot uniquely determine the corresponding distribution of forces.

Since the contact spot possesses radial symmetry, it seems reasonable to assume that the distribution of normal contact stresses σ_0 at rest also has this property.

In the next section we propose a simple analytic representation for the contact stresses distributions σ_0 assuming its radial symmetry

$$\sigma_0(x, y) \equiv \sigma_0(\sqrt{x^2 + y^2}) \qquad (2)$$

After the body is set in motion, there occur tangent stresses which "deform" the originally symmetric distribution of stresses by sort of "shifting" the whole picture in the direction of the instantaneous sliding velocity v or in the direction of rolling.

This "shift of symmetry" can be modelled by introducing a factor linear in coordinates x and y in the following way [5,6,8,10]:

$$\sigma(x, y) = \sigma_0(x, y)(1 + k_x x/R + k_y y/R) \qquad (3)$$

2.2 Integral dry friction models and its analytical approximations

Integrating the differentials (1) over the contact spot yields the resultant vector $\mathbf{F}$ of the friction force and torque $\mathbf{M}_\mathbf{C}$:

$$\mathbf{F} = -f\iint_G \sigma(x, y)\frac{\mathbf{V}}{|\mathbf{V}|}dxdy, \quad \mathbf{M}_\mathbf{C} = -f\iint_G \sigma(x, y)\frac{\mathbf{r}\times\mathbf{V}}{|\mathbf{V}|}dxdy, \qquad (4)$$
$$G = \{(x, y) : x^2 + y^2 \le R^2\},\, \mathbf{F} = (F_{||}, F_\perp)$$

After normalization of the variables x and y by using the characteristic size equal to the radius of the contact area R: $x = \hat{x}R,\, y = \hat{y}R$ and introduction of the dimensionless distribution function $\sigma(\hat{x}, \hat{y}) = \hat{\sigma}(\hat{x}, \hat{y})N/R^2$ the components (4) can be rewritten as [17]:

$$F_{\parallel} = f\iint_G \frac{(v-\omega y)(1+k_y\, y/R)\sigma_0 dxdy}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega vy}},\ F_{\perp} = \frac{k_x f}{R}\iint_G \frac{\omega x^2\sigma_0 dxdy}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega vy}}$$
$$M_C = f\iint_G \frac{(\omega(x^2+y^2)-vy)(1+k_y\, y/R)}{\sqrt{\omega^2(x^2+y^2)+v^2-2\omega vy}}\sigma_0 dxdy,\quad G=\{(x,y): x^2+y^2\le 1\} \tag{5}$$

Here the symbol ^ is omitted for brevity.

In applications, in the case of radial symmetry of the contact area spinning distorts the originally symmetric distribution of stresses to a considerably lesser extent than so do rolling and sliding. Therefore, it seems reasonable to assume $k_x \ll k_y$ in (3) and thereby neglect the corresponding terms in (5).

The exact integral models (4-5) give a good description of the dry friction effects in the case of combined kinematics. However, it is inconvenient to use them straightforwardly as it implies dealing with multiple integrals in the right-hand sides of the equations of motion. Approximated analytical models [4-13, 17] can help to avoid this inconvenience:

$$F_{\parallel} = \frac{F_0 v}{\sqrt{v^2+au^2}},\quad F_{\perp} = \frac{\mu k_x u}{\sqrt{u^2+bv^2}},\quad M_C = \frac{M_0 u}{\sqrt{u^2+mv^2}},\ u\equiv \omega R \tag{6}$$

The coefficients in (6) can be found as follows [5-6,8]:

$$F_0 = F_{\parallel}(0,v) = 2\pi f R^2\int_0^1 r\sigma_0(r)dr,\quad \frac{1}{\sqrt{a}} = \frac{u}{F_0}\frac{\partial F_{\parallel}(u,0)}{\partial v} = \frac{\pi f R^2}{F_0}\int_0^1 \sigma_0(r)dr,$$
$$\mu = \frac{1}{k_x}F_{\perp}(u,0) = \pi f R^3\int_0^1 r^2\sigma_0(r)dr,\quad \frac{\mu}{\sqrt{b}} = \frac{v}{k_x}\frac{\partial F_{\perp}(0,v)}{\partial v} = \pi f R^3\int_0^1 r^3\sigma_0(r)dr \tag{7}$$
$$M_0 = M_C(u,0) = 2\pi f R^3\int_0^1 r^2\sigma_0(r)dr,\quad \frac{1}{\sqrt{m}} = \frac{v}{M_0}\frac{\partial M_C(0,v)}{\partial u} = \frac{\pi f R^3}{M_0}\int_0^1 r^3\sigma_0(r)dr$$

In what follows we inject a certain randomness into the expression of the distribution σ_0 and show how this affects the parameters of the model (7).

2.3 Probabilistic model of friction

Assuming the distribution of the normal stresses to contain a certain random ingredient we however suppose that when the body is at rest the graph of σ_0 has the form qualitatively depicted in Fig. 2. Such a distribution was obtained numerically when studying the rolling motion of an aviation pneumatic tire with radial construction [5-8, 18]. As a first approximation we use a combination of the Hertz and Galin distributions of the form:

$$\sigma_0(r) = \left(\frac{1}{\sqrt{R^2-r^2}} - \frac{\sqrt{R^2-r^2}}{\xi^2-R^2}\right)\cdot\frac{3P_0\left(R^2-\xi^2\right)}{2\pi R\left(4R^2-\xi^2\right)}. \tag{8}$$

Here P_0 is the force of normal pressure or simply the body's weight and $\pm\xi$ are the coordinates of the points at which the distribution in Fig.2 has minima. The variable ξ is assumed to be a stochastic function of time.

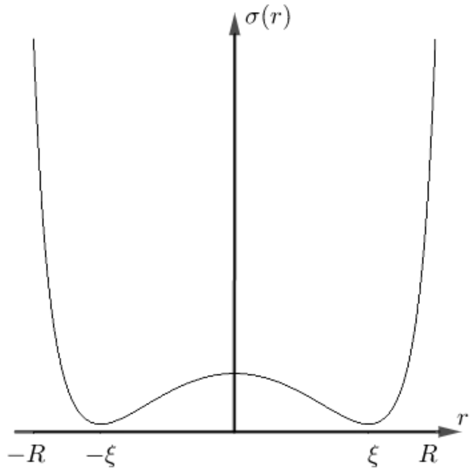

Figure 2: Typical distribution of the normal contact stresses for radial aviation pneumatic tire

To obtain explicit expressions of the force and moment one has to insert the formula (8) for $\sigma_0(r)$ into (5). The integrals that ensue can be calculated analytically but the final formulas are still too bulky.

From the point of view of engineer applications it seems to be effective to use the approximate model (7) whose coefficients in terms of dimensional variables now read

$$
\begin{gathered}
F_0 = fP_0,\, M_0 = \frac{3f\pi R\left(5R^2 - 4\xi^2\right)}{16\left(4R^2 - 3\xi^2\right)},\, \mu = \frac{M_0}{2},\, b = \frac{m}{4} \\
a = \left(\frac{8R\left(4R^2 - 3\xi^2\right)}{3\pi\left(3R^2 - 2\xi^2\right)}\right)^2, \quad m = \left(\frac{15\pi\left(5R^2 - 4\xi^2\right)}{16R\left(6R^2 - 5\xi^2\right)}\right)^2
\end{gathered}
\tag{9}
$$

The graphs of the exact (5) and approximate (6) component $F_{\|}$ and the moment M_C as functions of $k = v/u$ are shown in (Fig. 3). The graph of the normal component resembles (at least qualitatively) the graph of M_C.

3 CONCLUSIONS

We offered a model of combined dry friction that accounts for randomness in the distribution of the normal stresses within the contact spot.

For a typical distribution of stresses for a pneumatic aviation tire with radial construction [5-8, 18] the model's coefficients are calculated.

The graphs plotted show a good agreement between the friction force and moment components derived from the exact integral representation (5) and their analytical approximations (6).

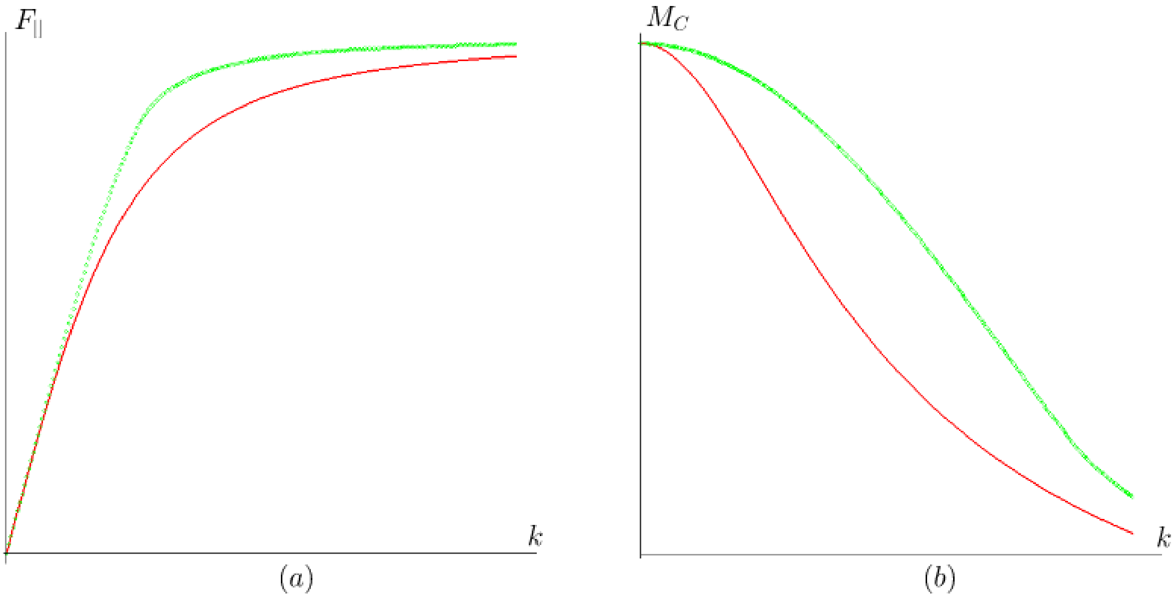

Figure 3: The exact force $F_{||}$ (a) and torque M_C (b) (green lines) and their approximations (red lines)

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REFERENCES

[1] Burlakov, D., Treschev, D. A Rigid Body on a Surface with Random Roughness. *Regular and Chaotic Dynamics* (2014) **19**: 296-309.
[2] Ivanov, A. On Singular Points of Equations of Mechanics. *Doklady Mathematics* (2018) **97(2)**:167-169.
[3] Ivanov, A.P. On the Equilibrium of Magic Stones. *Mechanics of Solids* (2018) **53(2)**: 26-31.
[4] Ivanov A.P.: A dynamically consistent model of the contact stresses in the plane motion of a rigid body, *Prikl. Mat. Mekh.*, (2009). **73(2)**: 134-144.
[5] Kireenkov, A.A. Improved theory of the combined dry friction in problems of aviation pneumatics dynamics. *Proceedings of the 7th International Conference on Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2017* (2017):1293-1298.
[6] Kireenkov, A.A. Improved friction model of the aviation tyre contact with the landing strip. *IFAC-PapersOnLine* (2018) **51(2)**:890-894. (DOI 10.1016/j.ifacol.2018.04.027)
[7] Kireenkov, A.A., Nushtaev, D.V., Zhavoronok, S.I. A new approximate model of tyre accounting for both deformed state and dry friction forces in the contact spot on the background of the coupled model. *MATEC Web of Conferences* (2018) **211(08003)**. (DOI 10.1051/matecconf/201821108003)
[8] Kireenkov, A.A., Zhavoronok, S.I. Coupled dry friction models in problems of aviation

pneumatics dynamics. *International Journal of Mechanical Sciences* (2017) **127**:198-203. (DOI 10.1016/j.ijmecsci.2017.02.004)

[9] Kireenkov, A.A., Zhavoronok, S.I. Implementation of analytical models of the anisotropic combined dry friction in problems of pneumatics' dynamics. *MATEC Web of Conferences* (2018) **211(08004)**. (DOI 10.1051/matecconf/201821108004)

[10] Kireenkov, A.A. Further development of the theory of multicomponent dry friction. *Proceedings of the 6th International Conference on Coupled Problems in Science and Engineering*, *COUPLED PROBLEMS 2015* (2015):203-209.

[11] Kireenkov A.A. and Semendyaev S.V. and Filatov V.V. Experimental Study of Coupled Two-Dimensional Models of Sliding and Spinning Friction. *Izv. Akad. Nauk. Mekh. Tverd. Tela*. (2010) **6**:192-202

[12] Kireenkov, A.A. Generalized two dimensional model of the sliding and spinning friction. *Doklady Physics*. (2010), **55(4)**:482—486.

[13] Kireenkov, A.A.: Coupled Models of Sliding and Rolling Friction. *Doklady Akademii Nauk*, (2008). **419(6)**: 759–762.

[14] Kireenkov, A.A. Three-dimensional model of combined dry friction and its application in non-holonomic mechanics. *Proceedings of ENOC-2005, Eindhoven, Netherlands, August 7-12* (2005):571-577.

[15] Kireenkov, A.A. On the Motion of a Homogeneous Rotating Disk along a Plane in the Case of Combined Friction. *Izv. Akad. Nauk. Mekh. Tverd. Tela*. (2002) **1**:60–67.

[16] Ramodanov, S.M., Kireenkov, A.A. Controllability of a rigid body in a perfect fluid in the presence of friction. *Proceedings of the 7th International Conference on Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2017* (2017):179 – 184.

[17] Semendyaev, S.V. Solid system with two massive eccentrics on a rough plane: rotational case. *IFAC-PapersOnLine* (2018) **51**(2):884-889.

[18] Zhavoronok, S.I., Kireenkov, A.A. On the effect of the anisotropic dry friction and the deformed state of tires on the shimmy initiation. *Proceedings of the 7th International Conference on Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2017* (2017):216-226.